# Two-component sandpile model : self-organized criticality of the second kind

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Two-component sandpile model is investigated numerically and theoretically. Monte Calro simulations have been performed to show that both avalanche size and lifetime obey the power-law distribution whose exponents are approximately equal to 1.5 and 2.0, respectively. This result indicates that the system exhibits SOC. A mean-field theory has also been considered to understand the origin of SOC in the model. We find that the system approaches a steady critical state and its universality class is different from that of existing one-component sandpile models. This reason comes from the conservation law of the numbers of two different sands in the local toppling rule which enables to coexist an infinite number of stable states which plays a role of dissipation. Instead of double control parameters in one-component sandpile models, a rate constant of dissipation is removed in two-component models. Consequently, we find a novel SOC model and its different class in the sense that the more conserved quantities result in the less control parameters.

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## I. INTRODUCTION

According to a vast amount of studies on fractals, $1/f$ noise, and so on, a lot of non-equiliblium systems in nature reportedly exhibit a critical state where the power-law distribution is observed. However, it has not been deeply understood why a huge variety of distributions spontaneously comes to the critical state. Self-organized criticality (SOC) proposed by Bak *et al.* [1–4] has provided a reasonable understanding for the emergence of criticality using various sandpile models. These models are cellular automata to move sands with threshold dynamics governed by local toppling rule. The domino effect of toppling sands to release local stresses in the system induces a sequence of topplings called an avalanche, whose size and lifetime distributions obey the power-law distribution without tuning any parameter. A number of works on SOC have been done numerically. Several analytical results have also been obtained. Dhar [5, 6] has found an exact solution in the abelian sandpile model and has shed light on the static behavior, such as the number of total recurrent states and hight correlation functions. However, its dynamics is still obscure, that is, the power-law distributions of avalanches still have not been derived analytically.

Though the sandpile models have no control parameters by appearance, Vespegnani and Zapperi [7, 8] have found that they do have hidden control parameters: a slow dissipation rate and a slower addition rate of sands. To explain this, they proposed a mean-field theory to consider the state equations, *i.e.*,

$$\frac{\partial}{\partial t}\rho_\kappa = f_\kappa(\rho_a, \rho_c, \rho_s), \qquad (\kappa = a, c, s), \tag{1}$$

where $\rho_a$, $\rho_c$, and $\rho_s$ are the state densities of active, critical, and stable states, respectively. Imposing the conservation law of the number of sands given by the local toppling rule, it is concluded that the models become critical in the double limit, *i.e.*,

$$\epsilon, \delta \to 0, \qquad \rho_a = \delta/\epsilon \to 0, \tag{2}$$

where $\epsilon$ is a dissipation rate and $\delta$ an addition rate. In other words, $\rho_a$ is an order parameter and $\epsilon$, $\delta$ are control parameters and SOC is achieved by rough tuning of the *unapparent* parameters $\epsilon$, $\delta$ around zero. It is found that most of SOC models [9, 10] belong to the same *universality class.* We call this class *SOC of the first kind.* It seems interesting whether other kind of SOC exists, or equivalently, whether the condition of the double limit (2) are strictly the necessary condition of SOC ? In this letter, we give an answer to these questions.

Tsuchiya and Katori [11] have proved rigorously that SOC breaks down in the abelian sandpile model when the toppling rule of sands is non-conservative. Here, we pay attention to the number of conservation laws. Does anything new happen if the number of conservation laws is increased? How about the robustness of SOC? To answer these questions, we consider *two-component sandpile models* which deal with two kinds of sands. This means the models have two kinds of conservation laws. Firstly, numerical simulations are carried out to check power-law behavior. Secondly, a mean-field theory is investigated to examine the essence of the model.

## II. MODEL DESCRIPTION

Two-component sandpile model is a cellular automaton defined on a regular lattice. A pair of two non-negative integers, $\boldsymbol{h}(\boldsymbol{x}) \equiv (i, j)$, is assigned to each site, $\boldsymbol{x}$, on the lattice, where $i, j(\geq 0)$ denote the numbers of sand $A$ and $B$. At each time step, one unit of sand $A$ or $B$ is added to a randomly chosen site at a relative ratio $(0 <)r_{AB}(< 1)$ of sand $A$ to sand $B$. We can consider several types of toppling rule for the time evolution of the system. For example, the toppling occurs when

(a) $i \geq i_{th}$ *or* $j \geq j_{th}$,

(b) $i \geq i_{th}$ *and* $j \geq j_{th}$, or

(c) $i + j \geq k_{th}$ $(i, j > 0)$ where $i_{th}$, $j_{th}$,

where $k_{th}$ is a certain threshold value. If the number of sands at a site reaches threshold, sand $A$ and $B$ on the site topple one by one to randomly chosen nearest neighbor sites until the number of sands at the site becomes less than the threshold. In these rules, sand $A$ and $B$ topple jointly. For example, if a site has $n$ grains of sands $A$ and no sands $B$, and then $m$ $(\leq n)$ new grains of sands $B$ are added, it ends up with $n-m$ grains of sands A and no sands $B$. These rules satisfy two kinds of conservation laws of local toppling. Avalanche continues unless the numbers of sands in all the sites becomes less than the threshold.

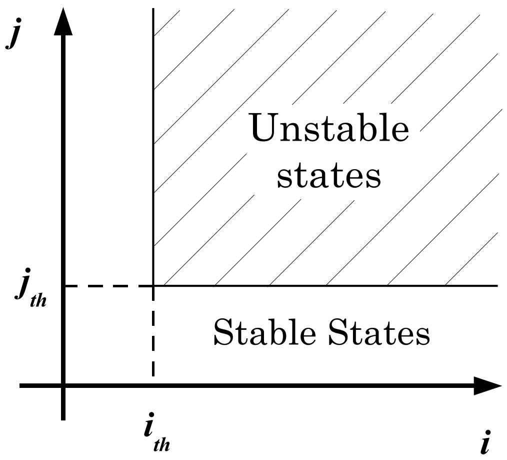


FIG. 1: Stable and unstable states in the rule (b).

Among three rules of toppling, the *AND* rule (b) is the most interesting because of the presence of *infinite* stable states as shown in Fig. 1.

Stable states $\boldsymbol{h}_s$ and unstable states $\boldsymbol{h}_u$ are defined by

$$\boldsymbol{h}_s = \{(i,j) \mid 0 \le i < i_{th} \text{ or } 0 \le j < j_{th}\}, \tag{3}$$

$$\boldsymbol{h}_u = \{(i,j) \mid i \ge i_{th} \text{ and } j \ge j_{th}\}. \tag{4}$$

The stable states possibly absorb all the sands added into the system. This means that all avalanches inevitably stop without introducing artificial boundary dissipation, which is indispensable in all one-component sandpile models. Rules (a) and (c) cannot remove the boundary dissipation because the number of stable states is finite under these rules. Hereafter, we consider the rule (b) with periodic boundary conditions and the threshold is fixed to $i_{th} = j_{th} = 1$.

## III. NUMERICAL RESULTS

First, we numerically investigate avalanche size $S$ and its lifetime $T$. Here, $S$ is defined by the total number of topplings in an avalanch, and $T$ is done by the total time steps in an avalanche. The cumulative distribution function (CDF) of avalanche size, $D(S)$, and lifetime, $D(T)$, in the one-dimensional lattice are calculated by numerical simulations and they are shown in Figs. 2. We can observed that both CDFs obey the power-laws distirubtion,

$$D(S) \sim S^{-\tau}, \quad D(T) \sim T^{-\alpha}, \tag{5}$$

where the power-law exponents are approximately $\tau \simeq 0.5$ and $\alpha \simeq 1.0$. Equivalently, the probability distribution function (PDF) is drived by

$$P(S) \sim S^{-(1+\tau)}, \quad P(T) \sim S^{-(1+\alpha)}. \tag{6}$$

The exponents in PDF becomes $1 + \tau \simeq 1.5$ and $1 + \alpha \simeq 2.0$. They are found to be very close to the critical exponents of branching processes [12]. We carried out numerical simulations in $2-$ and $3-$dimension and obtained almost the same distributions in CDF and PDF. These results imply the model has meanfield-like characteristics [13]. We also find that varying $r_{AB}$ from 0.5 to 0.1 does not affect the power-law tails. This means that the model ends up with spontaneously going to a SOC state robustly in the long time limit without tuning $r_{AB}$. It becomes evident that the two-component sandpile model exhibits SOC. It should be noted that because of the absence of boundary dissipation, the power-law tails become extended infinitely as the number of added sands $N$ increases even in a finite system.

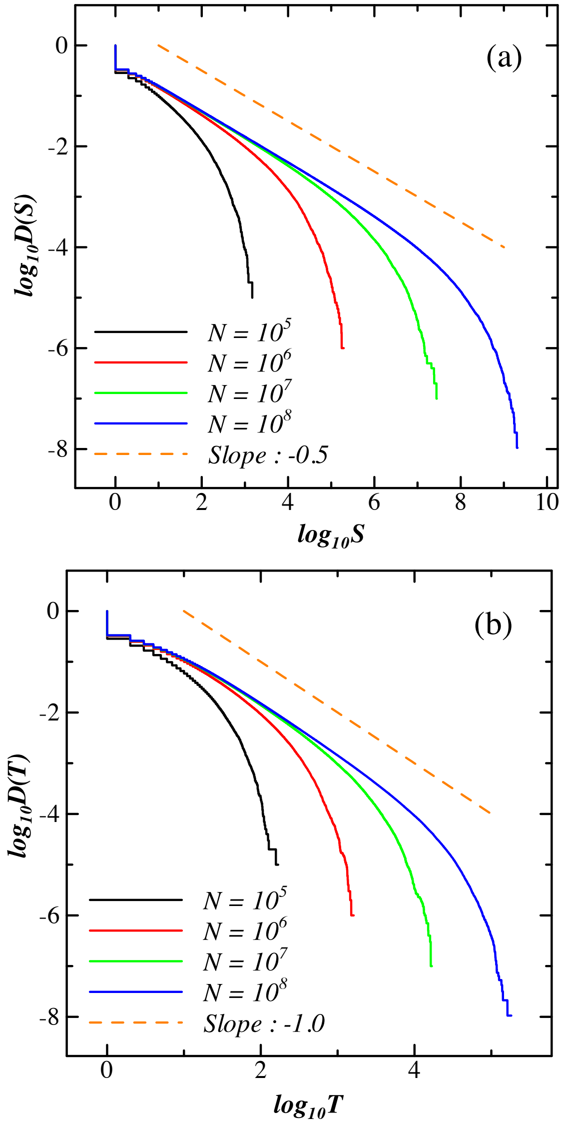


FIG. 2: Double logarithmic plots of CDFs of avalanche size (a) and lifetime (b). The lattice size $L = 10^4$ and the number of added sands $N = 10^5$, $10^6$, $10^7$, and $10^8$, respectively. Sands $A$ and $B$ are added at equal rate $r_{AB} = 0.5$.

## IV. MEAN-FIELD THEORY AND ITS RESULTS

Next, we construct a mean-field theory of the two-component sandpile model and compare to that of the one-component models [7, 8]. We introduce sets of probability densities in stable states, $\boldsymbol{X}$, and unstable states, $\boldsymbol{Z}$, as

$$\boldsymbol{X} = \{X_{(i,j)} \mid 0 \leq i < 1 \text{ or } 0 \leq j < 1\}, \tag{7}$$

$$\boldsymbol{Z} = \{Z_{(i,j)} \mid i \geq 1 \text{ and } j \geq 1\}. \tag{8}$$

Rate equations for $\boldsymbol{X}$ and $\boldsymbol{Z}$ give an infinite system of first-order nonlinear ordinary differential equations. However, it is difficult to handle the infinite degrees of freedom. To avoid this difficulty, we define the following reduced six variables.

$$\boldsymbol{X}^* = \{X_0, X_A, X_B\}, \qquad \boldsymbol{Z}^* = \{Z_0, Z_A, Z_B\} \tag{9}$$

with

$$\begin{aligned} X_0 &\equiv X_{(0,0)}, & Z_0 &\equiv \sum_{i=1}^{\infty} Z_{(i,i)}, \\ X_A &\equiv \sum_{i=1}^{\infty} X_{(i,0)}, & Z_A &\equiv \sum_{i=2}^{\infty} \sum_{j=1}^{\infty} Z_{(i,j)}, \\ X_B &\equiv \sum_{j=1}^{\infty} X_{(0,j)}, & Z_B &\equiv \sum_{i=1}^{\infty} \sum_{j=2}^{\infty} Z_{(i,j)}. \end{aligned}$$

For obtaining a closed set of equations for $\boldsymbol{X}^*$ and $\boldsymbol{Z}^*$, we assume the Poisson distribution for $X_{(i,0)}$ and $X_{(0,j)}$ as

$$\frac{X_{(i,0)}}{X_A} = \frac{\mu_A^{i-1}}{(i-1)!}e^{-\mu_A} \quad (i \geq 1), \tag{10}$$

$$\frac{X_{(0,j)}}{X_B} = \frac{\mu_B^{j-1}}{(j-1)!}e^{-\mu_B} \quad (j \geq 1) \tag{11}$$

where $\mu_A \equiv \delta_A t$, $\mu_B \equiv \delta_B t$ are the mean values of each Poisson distribution at time $t$. We also define $\delta_A$ and $\delta_B$ as the rate constants of additions of each sand. Therefore, the rate equations for the reduced variables are described by

$$\begin{aligned}
\frac{dX_0}{dt} &= -2X_0(Z_0 + Z_A + Z_B) + Z_0 \\
&\quad -(\delta_A + \delta_B)X_0, && (12)\\
\frac{dX_A}{dt} &= (X_0 - X_A)(Z_0 + Z_A + Z_B) + Z_A \\
&\quad +\delta_A X_0 - \delta_B X_A, && (13)\\
\frac{dX_B}{dt} &= (X_0 - X_B)(Z_0 + Z_A + Z_B) + Z_B \\
&\quad +\delta_B X_0 - \delta_A X_B, && (14)\\
\frac{dZ_0}{dt} &= (\alpha_A X_A + \alpha_B X_B - 2Z_0)(Z_0 + Z_A + Z_B) - Z_0 \\
&\quad +\alpha_B \delta_A X_B + \alpha_A \delta_B X_A, && (15)\\
\frac{dZ_A}{dt} &= \{(1-\alpha_A)X_A + Z_0\}(Z_0 + Z_A + Z_B) - Z_A \\
&\quad +(1-\alpha_A)\delta_B X_A, && (16)\\
\frac{dZ_B}{dt} &= \{(1-\alpha_B)X_B + Z_0\}(Z_0 + Z_A + Z_B) - Z_B \\
&\quad +(1-\alpha_B)\delta_A X_B, && (17)
\end{aligned}$$

where $\alpha_A(t) \equiv X_{(1,0)}/X_A = \exp(-\delta_A t)$ and $\alpha_B(t) \equiv X_{(0,1)}/X_B = \exp(-\delta_B t)$.

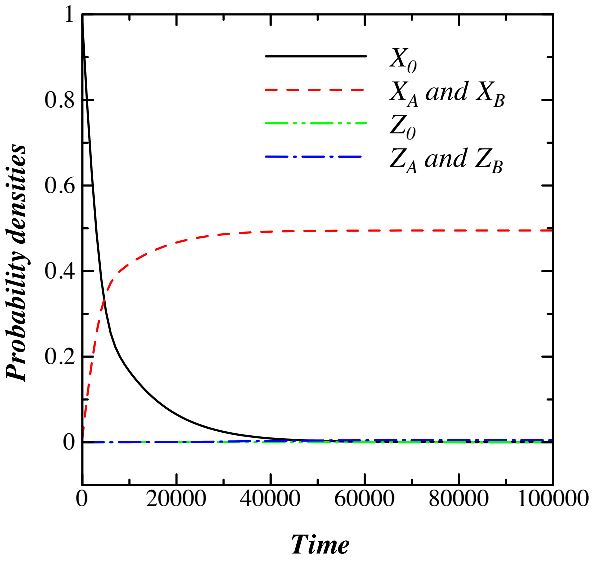


FIG. 3: Time evolution of the reduced six variables at $\delta_A = \delta_B = 0.0001$. Initial values are set to $X_0 = 1.0$ and $X_A = X_B = Z_0 = Z_A = Z_B = 0.0$. At the steady state, $X_0 = Z_0 \simeq 0$, $X_A = X_B \simeq 0.5$, and $Z_A = Z_B \simeq \sqrt{\delta_A}/2 = 0.005$.

A numerical solution for the equation system (12)-(17) are plotted in Fig. 3. We find that in the long time limit, the system goes to a steady state which is supposed to be SOC. The steady state can be examined analytically. When $t \to \infty$, then $\alpha_A, \alpha_B \to 0$ and $X_0, Z_0 \to 0$. Therefore, the following relations are derived from Eqs. (12)-(17).

$$X_A = \frac{Z_A}{Z_A + Z_B + \delta_B}, \qquad X_B = \frac{Z_B}{Z_A + Z_B + \delta_A}. \tag{18}$$

Suppose that $Z_A + Z_B \gg \delta_A, \delta_B$ and $Z_A : Z_B = \delta_A : \delta_B$, we obtain

$$Z_A = \frac{\sqrt{2}\delta_B^{\frac{1}{2}}\delta_A^{\frac{3}{2}}}{(\delta_A + \delta_B)^{\frac{3}{2}}}, \quad Z_B = \frac{\sqrt{2}\delta_A^{\frac{1}{2}}\delta_B^{\frac{3}{2}}}{(\delta_A + \delta_B)^{\frac{3}{2}}}. \tag{19}$$

Equations (19) show that the model goes to a critical state ($Z_A = Z_B = 0$) in the limit $\delta_A, \delta_B \sim \delta \to 0$. Out of two conditions (2) for SOC of the first kind, our model can successfully remove one of them with respect to the control parameter of dissipation $\epsilon$. Furthermore, the order parameters behave as $Z_A, Z_B \sim \delta^{1/2}$ when the control parameters go to zero $\delta \to 0$. These two characteristic features of the two-component model, that is, the existence of the steady state without dissipation and the asymptotic behavior to the critical state in $\delta \to 0$, distinctively differ from conventional SOC models. This indicates a strong evidence that the system belongs to a different universality class. Consequently, we are successfully able to find the new mechanism and class of SOC, which we call *SOC of the second kind.*

## V. DISCUSSION

We have investigated two-component sandpile models with the AND rule (b) with periodic boundary conditions mainly on $1-$dimensional lattices. Extention of the sandpile model to two-component one is essential to generate an infinite number of stable states which substitute for boundary dissipation. Therefore, the model becomes more natural because the dissipation is usually introduced artificially only for stopping avahanches. Simulation results indicate the system exhibits SOC.

The mean-field theory confirms that the model goes to the steady and critical state belonging to the different universality class from existing one-component models. Here, only rough-tuning of rate constants of additions $\delta$ is enough to induce SOC. We can successfully construct the new SOC where the system does not have any artificial dissipation mechanism. Consequently, the two conditions (2) are not the necessary condition for SOC, which is the answer to the question mentioned previously.

It is interesting to view SOC models from the standpoint of the number of conserved quantities. Whe the model has no conserved quantities, such as contact processes, show the power-law distribution strictly at the critical point. Therefore, fine-tuning of control parameters is required. When the model has one conserved quantity, such as one-component sandpile models, they become SOC in rough-tuning of *two* control parameters, an addition and a dissipation rates of sands, around zero. The two-component sandpile model with two conserved quantities show SOC in rough-tuning of only *one* control parameter, an addition rate of sands, around zero. It could be concluded that the more conservation laws result in the less control parameters.

Sandpile models are considered to have some relevance to power laws of earthequake magnitude distributions, called Guthenberg-Richter law [14]. One-component models have to introduce some artificial boundary dissipations to stop the avalanche. However, the surface of the earth has no apparent boundary. At this point, it is more natural to apply two-component sandpile models to earthequake magnitude distributions. Furthermore, an earthquake could be triggered by multiple physical quantities, such as elastic energy and stress of earth's crust. If an earthquake takes place only when both the energy and stress reach the threshold simultaneously, the distribution of earthquakes would be well described by the SOC of the second kind. In this paper, we express the process in terms of a sandpile. Obviously, the sand is merely a symbol and it could be any kind of substances which trigger the same dynamics. In addition to earthquakes, therefore, a wide variety of SOC phenomena is triggered by multiple factors and it would belong to SOC of the second kind. This point should be made clear for future work.

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